Risk Measures and Interest Rate Ambiguity

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Introduction and Motivation

Risk management

- New developments, new regulatory rules.
- Credit risk regulation: Bâle II

**Question**: How to take into account ambiguity on interest rates, or time to default? Classical monetary risk measures are not adapted tools to do that.

**Design of new financial products in this context**

- Credit derivatives
- How to design financial contracts when ambiguity on time to default.
- Which optimal risk transfer in this context?

**Main question**

How to separate the specific risk of the future exposure of the discount (defaутable) risk
Some related works (among many others...!!)

* Insurance literature on optimal policy design: Borch (1962), Bühlmann (1970), Raviv (1979), Gerber (1980)...


* Risk measures
Convex Risk measures : Basic properties

Definition: Let \((\Omega, \mathcal{F})\) be a standard measurable space and \(\mathcal{X}\) the linear space of bounded functions (including constant functions).

The functional \(\rho\) is a \textbf{monetary risk measure} if it satisfies:

- **Convexity, and Decreasing monotonicity;**
- **Translation invariance:** \(\forall X \in \mathcal{X}, \forall m \in \mathbb{R}, \quad \rho(X + m) = \rho(X) - m.\)
  
  \(\text{In particular, } \rho(X + \rho(X)) = 0.\)

**Dual representation:** The convexity of the framework leads to an "explicit" representation. There exists a \textbf{penalty function} \(\alpha\) taking values in \(\mathbb{R} \cup \{+\infty\}\) such that:

\[
\forall \Psi \in \mathcal{X}, \quad \rho(\Psi) = \sup_{Q \in \mathcal{M}_{1,f}} \{\mathbb{E}_Q[-\Psi] - \alpha(Q)\}
\]

\[
\forall Q \in \mathcal{M}_{1,f}, \quad \alpha(Q) = \sup_{\Psi \in \mathcal{X}} \{\mathbb{E}_Q[-\Psi] - \rho(\Psi)\}
\]

where \(\mathcal{M}_{1,f}\) is the set of all additive measures on \((\Omega, \mathcal{F})\).

Moreover, the supremum is attained in the first equation in \(\mathcal{M}_{1,f}\).
Cash-Invariance and Discounting

Definition of monetary risk measure implicitly assumes that future risk position and risk measure are expressed in the same numéraire.

Convention : $1_T = 1$ unit of cash available at time $T$, and $D_{0,T}$ is a random discount factor.

Spot risk measure (Foellmer-Schied, 2004) et Cash-invariance

$$\rho_0(D_{0,T}X_T + m_{10}) = \rho_0(D_{0,T}X_T) + \rho_0(m_{10}), \quad \rho_0(m_{10}) = -m_{10}$$

Forward risk measure (Rouge-El Karoui, 2000))

$$\rho_T(X_T) = \rho_T(X_T + m_{1T}) = \rho_T(X_T) + \rho_T(m_{1T}) \text{ and } \rho_T(1_T) = -1$$
Forward versus Spot risk measure

Suppose a zero-coupon bond with maturity $T$ is traded on the market at $B(0, T)$. As in interest rates framework, define

$$\mathcal{R}_0(D_{0,T}X_T) := B(0, T) \rho_T(X_T)$$

- Then, $\mathcal{R}_0$ is a cash-invariance risk measure iff $\rho_T$ satisfies the calibration constraint: $\rho_T(\lambda D_{0,T}^{-1}) = -\lambda B(0, T)^{-1}$. Then, for any $Q_T$ in the domain of $\alpha_T$, $E_{Q_T}(D_{0,T}^{-1}) = B(0, T)^{-1}$.
- Characterisation of $\mathcal{R}_0$ domain: $Q_0$ belongs to dom($\alpha_0$) iff the forward measure $Q_T$ defined as usual by $\frac{dQ_T}{dQ_0} = D_{0,T}/B(0, T)$ belongs to dom($\alpha_T$).

What happens when zero-coupon bonds are non quoted on the market, and more generally when ambiguity on interest rates occurs.
Cash sub-additive Risk measure

**Motivation**: Given a risk measure $\rho$ and a discount factor $D_{0,T} \in [0, 1]$, observe that, if $\mathcal{R}(X_T) = \rho(X_T D_{0,T})$, then for $\forall X_T \in \mathcal{X}_T$

$$\forall m \geq 0, \quad \mathcal{R}(X_T + m1_T) = \rho(X_T D_{0,T} + mD_{0,T})$$

$$\geq \rho(X_T D_{0,T} + m1_0) = \rho(X_T D_{0,T}) - m1_0 = \mathcal{R}(X_T) - m1_0.$$ 

We take this property as the natural one to replace the cash invariant axiom, for risk functional expressed as *today cash* but directly defined of future position.

**Def**: Any convex, decreasing, functional $\mathcal{R}$ defined on $\mathcal{X}_T$ is called Cash sub-additive Risk measure iff for any $X_T \in \mathcal{X}_T$ the function $m \in \mathcal{R} \mapsto \mathcal{R}(X_T + m1_T) + m1_0$ is nondecreasing.
Cash sub-additive Risk measure

Example: Ambiguous discount factors

- Assume that the *classical risk measure* used by the regulator is $\bar{\rho}$, a spot-cash invariant risk measure on $\mathcal{X}_T$. But in fact only a *confidence interval* for the discount factors is available, that is unknown $D_T$ are ranging between two constants: $0 \leq d_L \leq d_H \leq 1$.

- To give a price to discount factor ambiguity, Regulator assesses risk of future payoff $X_T$ in the **worst case interest rates scenario**:

$$
R^{\bar{\rho}, D}(X_T) := \sup \{ \bar{\rho}(D_T X_T) \mid D_T \in \mathcal{X}, \quad 0 \leq d_L \leq D_T \leq d_H \leq 1 \}.
$$

$R^{\bar{\rho}, D}$ is obviously a **cash sub-additive risk measure**. Moreover,

$$
R^{\bar{\rho}, D}(X_T) = \bar{\rho}(-v(X_T)), \quad v(x) = -(d_L x^+ - d_H (-x)^+)
$$

- The classical measure of risk used in global risk management is the **Put premium** risk measure (Jarrow (2002)), defined as the expected loss

$$
R(X_T) = \frac{1}{r} \mathbb{E}[(X_T)^+], \quad r \geq 1.
$$

- Same result still holds, if $d_H$ and $d_L$ are given $\mathcal{F}_T$-r.v in $[0, 1]$. 

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Proposition

- Let \( v_T \) be a \( \mathcal{F}_T \) family of real, convex, decreasing functions such that: \( v_T(0) = 0 \), and \( (v_T)'_x \in [-1, 0] \). Let \( \beta_T(y) = \sup_{x \in \mathbb{R}} \{xy - v(x)\} \) be the \( \mathcal{F}_T \)-measurable convex convex Fenchel transform of \( v_T \), defined on \([-1,0]\). For instance, for ambiguous discount factors, \( \beta_T(y) = l(-y) \) where \( l \) is the convex indicator function of the random interval \( \mathcal{D}_T = \{D_T \in \mathcal{X}_T|0 \leq D_L \leq D_T \leq D_H \leq 1\} \)

- \( \overline{\rho} \) is a cash invariant r.m. on \( \mathcal{X} \) with minimal penalty function \( \alpha \overline{\rho} \).

\[ \Rightarrow \mathcal{R}_{\overline{\rho},v}(X_T) := \overline{\rho}(-v(X_T)) \text{ is a cash sub-additive r.m.}, \text{ such that} \]

\[ \mathcal{R}_{\overline{\rho},v}(X_T) = \sup_{D_T \in \mathcal{X}} \{\overline{\rho}(D_T X_T + \beta(-D_T)) \mid 0 \leq D_T \leq 1\} \]

\[ \Rightarrow \text{Dual representation.} \]

\[ \mathcal{R}_{\overline{\rho},v}(X_T) = \sup_{\overline{Q}, D_T} \left\{ \mathbb{E}_{\overline{Q}}[-D_T X_T] - \alpha_{\overline{\rho},v}(\overline{Q}, D_T) \mid 0 \leq D_T \leq 1 \right\} \]

where \( \alpha_{\overline{\rho},v}(\overline{Q}, D_T) := \alpha \overline{\rho}(\overline{Q}) + \mathbb{E}_{\overline{Q}}[\beta_T(-D_T)] \)
Dual Representation

The dual representation can be obtained

- either using convex analysis tools
- or extending cash sub-additive r.m. to cash invariant r.m. and then exploiting their dual representation

We are looking for a “minimal” extension of $\mathcal{R}$, based on the key property: If $\hat{\rho}(X_T, x) := \mathcal{R}(X_T 1_T - x 1_T) - x$, then this functional is cash invariant as function of $(X_T, x)$,

$$
\hat{\rho}(X_T + m, x + m) = \mathcal{R}(X_T + m - (x + m)1_T) - (x + m) = \hat{\rho}(X_T, x) - m
$$
Minimal enlarged space

$(X_T, x)$ may be viewed as the “coordinates” of some r.v. $\hat{X}_T$ on the extended space $\hat{\Omega} = \Omega \times \{1, 0\}$ such that

$$\hat{X}_T(\omega, \theta) = X_T(\omega)1_{\{\theta = 1\}} + x1_{\{\theta = 0\}}, \quad \text{so that} \quad \{\theta = 0\} \text{is a atome}$$

The correspondence is one to one.

**Extended Risk measure**

Let $\mathcal{R}$ be a cash sub-additive risk measure defined on $\mathcal{X}_T$. $\mathcal{R}$ generates a cash-additive risk-measure $\hat{\rho}$ on the space $\hat{\mathcal{X}}_T$ of the bounded $\hat{\mathcal{F}}_T$-r.v.

$$\hat{\rho}(X_T1_{\{\theta = 1\}} + x1_{\{\theta = 0\}}) = \hat{\mathcal{R}}(X_T, x) := \mathcal{R}(X_T - x1_T) - x1_0,$$

$\hat{\rho}(X_T(\omega)1_{\{\theta = 1\}})$ can be identified with $\mathcal{R}(X_T)$.

$$\hat{\mathcal{R}}(X_T, x) = \mathcal{R}(X_T - x) - x \leq \mathcal{R}(X_T - y) - y \leq \hat{\mathcal{R}}(Y_T, y) - y = \hat{\mathcal{R}}(Y_T, y).$$

In terms of “economic default time”, $\{\theta = 1\}$ may be viewed as $\{T < \tau\}$.
Dual representation of $\mathcal{R}$

Any probability measure $\hat{\mathbb{Q}}$ on $\hat{\Omega}$ can be decomposed into

\[
\hat{\mathbb{Q}}(X_T \mathbf{1}_{\theta = 1} + x \mathbf{1}_{\theta = 1}) = \hat{\mathbb{Q}}(\{\theta = 1\})\hat{\mathbb{Q}}(X_T|\{\theta = 1\}) + x(1 - \hat{\mathbb{Q}}(\{\theta = 1\})) \\
= \hat{q} \mathbb{Q}^1(X_T) + x (1 - \hat{q}) = \mu(X_T) + x(1 - \mu(1))
\]

**Characterization** The minimal penalty function $\hat{\alpha}(\hat{\mathbb{Q}}) = \alpha(\mu)$ depends only on $\mu = \hat{q} \mathbb{Q}^1$, and

\[
\begin{align*}
\hat{\alpha}(\hat{\mathbb{Q}}) &= \sup_{X_T \in \mathcal{X}_T, x \in \mathbb{R}} \left\{ \mathbb{E}_{\hat{\mathbb{Q}}}[-(X_T - x)\mathbf{1}_{\theta = 1}] - \mathcal{R}(X_T - x) \right\} \\
&= \sup_{Y_T \in \mathcal{X}_T} \left\{ \mu(-Y_T) - \mathcal{R}(Y_T) \right\} \\
\mathcal{R}(X_T) &= \sup_{\mu \in \mathcal{M}_i} \left\{ \mu(-X_T) - \alpha(\mu) \right\}
\end{align*}
\]

**Remark** The risk measure $\hat{\mathcal{R}}$ is not defined on the space $\mathcal{X}_T$
More natural extension

The aim of this second extension is to define a risk measure \( \tilde{\rho} \) on a space \( \tilde{\mathcal{X}}_T \) of bounded r.v. containing \( \mathcal{X}_T \). We need

\( \Rightarrow \) to define the r.v. \( \tilde{X}_T = X^1_T 1\{\theta=1\} + X^0_T 1\{\theta=0\} \)

\( \Rightarrow \) to introduce a arbitrary(normalized) monetary risk measure \( \rho_0 \) on order to specify a “a priori” risk measure for \( X^0_T \).

Two monetary risk measures The functional \( \tilde{\rho} \) defined by

\[
\tilde{\rho}(X^1_T 1\{\theta=1\} + X^0_T 1\{\theta=0\}) = \mathcal{R} \left( X^1_T + \rho_0(X^0_T) \right) + \rho_0(X^0_T)
\]

is a monetary risk measure, with minimal penalty function, for any \( \tilde{Q} = \tilde{q}^1 \tilde{Q}^1 + \tilde{q}^0 \tilde{Q}^0 \)

\[
\tilde{\alpha}(\tilde{Q}) = \alpha_{\mathcal{R}}(\tilde{q}^1 \tilde{Q}^1) + \tilde{q}^0 \alpha_0(\tilde{Q}^0)
\]
The functional \( \rho_\mathcal{R}(X_T) = \tilde{\rho}(X_T, X_T) = \mathcal{R}(X_T + \rho_0(X_T)) + \rho_0(X_T) \) is a monetary risk measure with penalty functional

\[
\alpha_{\rho_\mathcal{R}}(Q) = \inf_{(\tilde{q}^1, \tilde{Q}^1, \tilde{q}^0, \tilde{Q}^0 | Q = \tilde{q}^1 \tilde{Q}^1 + \tilde{q}^0 \tilde{Q}^0)} \alpha_\mathcal{R}(\tilde{q}^1 \tilde{Q}^1) + \tilde{q}^0 \alpha_0(\tilde{Q}^0)
\]

- \( \rho_\mathcal{R} \) does not allow us to recover \( \mathcal{R} \).
- Given \( \tilde{Q} \in \text{Dom}(\tilde{\alpha}) \), and \( Q \) the induced probability measure on \( \mathcal{F}_T \). Then, \( Q \) belongs to \( \text{Dom}(\alpha_{\rho_\mathcal{R}}) \). Let us denoted by \( D_{0,T} \) the \( \tilde{Q} \)-conditional expectation of \( 1_{\theta=1} \) given \( \mathcal{F}_T \). Then, \( \tilde{Q}(X_T 1_{\theta=1}) = Q(X_T D_{0,T}) \), and

\[
\tilde{Q}(X_T 1_{\theta=0}) = Q(X_T (1 - D_{0,T})), \iff dQ^0 = \frac{(1 - D_{0,T})}{\tilde{q}^0} dQ
\]

\[
\mathcal{R}(X_T) = \sup_{(Q, D_{0,T}) \in \mathcal{A}} \{Q(-X_T D_{0,T}) - (\alpha_\mathcal{R}(D_{0,T}.Q) + \tilde{q}^0 \tilde{\alpha}(Q^0)) \}
\]

\[
\mathcal{A} = \{(Q, D_{0,T}) \text{s.t.} Q \in \text{Dom}(\alpha_{\rho_\mathcal{R}}), D_{0,T}.Q \in \text{Dom}(\alpha_\mathcal{R}), \tilde{Q}^0 \in \text{Dom}(\alpha_0)\}
\]
Conditional Risk measures

A natural question is to try to factorize $\tilde{\rho}$ in terms of $\rho_{\mathcal{R}}$ and some conditional risk measure between $\tilde{\mathcal{F}}_T$ and $\mathcal{F}_T$.

**Convex function and Risk measure on $\{0, 1\}$** Let $\rho$ be a monetary risk measure on $\lambda 1_{\{\theta=1\}} + \mu 1_{\{\theta=0\}}$. Then

$$\rho(\lambda 1_{\{\theta=1\}} + \mu 1_{\{\theta=0\}}) = \rho((\lambda - \mu) 1_{\{\theta=1\}}) - \mu$$

Put $v(x) = \rho(x 1_{\{\theta=1\}})$.

Then $v$ is a real convex decreasing function such that $v'(x) \geq -1$.

**Natural extension to previous filtrations**, with $\mathcal{F}_T$-random $V_T(X_T)$.

- Natural way to generate cash subadditive r.m : $\mathcal{R} \tilde{\rho},(X_T) := \tilde{\rho}(-V_T(X_T))$
- In general, such decomposition does not hold
Optimal Derivative Design

\[ Agent \ A \]
Exposure \( X^A_T \) \quad \text{F}_T \quad \text{Exposure} \( X^B_T \)

\[ \pi_0 \]

\[ \Rightarrow \] We want to determine the optimal transaction \((F, \pi)\).

Transaction feasibility

\[ \star \] Agent A looks for a hedge of her exposure: \( \inf_{F \in \mathcal{X}, \pi} \mathcal{R}_A(X^A_T - F_T) - \pi_0 \).

\[ \star \] Agent B wants to improve her risk measure: \( \mathcal{R}_B(X^B_T + F_T) + \pi_0 \leq \rho_B(X^B_T) \).

\[ \Rightarrow \] Optimal pricing rule: \( (\pi^*_B)_0(F_T) = \mathcal{R}_B(X^B_T) - \mathcal{R}_B(X^B_T + F) \).
Inf-convolution

**Theorem**: Let $\mathcal{R}_A$ and $\mathcal{R}_B$ be two cash sublinear risk measures with respective penalty functions $\alpha_A$ and $\alpha_B$. Let $\mathcal{R}_{A,B}$ be the inf-convolution of $\mathcal{R}_A$ and $\mathcal{R}_B$

$$\Psi \to \mathcal{R}_{A,B}(\Psi) \equiv \mathcal{R}_A \square \mathcal{R}_B(\Psi) = \inf_{H \in \mathcal{X}} \{ \mathcal{R}_A(\Psi - H) + \mathcal{R}_B(H) \}$$

and assume that $\mathcal{R}_{A,B}(0) > -\infty$.

- Then $\mathcal{R}_{A,B}$ is a cash sub-linear convex risk measure.
- The associated penalty function is given by

$$\forall \mu \in \mathcal{M}_1^s, \quad \alpha_{\mathcal{R}_{A,B}}(\mu) = \alpha_A(\mu) + \alpha_B(\mu).$$

- $\mathcal{R}_{A,B}$ is continuous from below as soon as this property holds for $\mathcal{R}_A$ or $\mathcal{R}_B$.
Infinitesimal Risk Measures and BSDE’s

Let \((\Omega, (\mathcal{F}_t), \mathbb{P})\) be the natural space associated with a Brownian motion \(W = (W_t, t \leq T)\). Following BSDE’s point of view (NEK-Peng-Quenez, Pengg-expectation, Barrieu-NEK, Schweitzer, Delbaen....), we define

\[-dY_t = g(t, Y_t, Z_t)dt - \langle Z_t, dW_t \rangle, \quad Y_T = -X_T\]

The solution is a pair of adapted processes \((\mathcal{R}_t^g, Z_t)\) in some \(L^2\) space.

- (Peng,Barrieu-NEK) if \(g\) does not depend on \(y\) and convex in \(z\), then \(\rho_t(X_T) = Y_t\) is a cash invariante convex risk measure (+ \(g\) with quadratic growth)

- Only assume \(g\) to be convex in both variables \((y,z)\), and decreasing w.r to \(y+\) (linear growth in \(y\) quadratic in \(z\)). Then \(\mathcal{R}_t^g(X) = Y_t\) is a cash sub-additive risk measure.

- The proof is based on : \(\mathcal{R}_t^g(X + m) + m = Y_t^m\) is solution of the same BSDE with coefficient \(g^m(t, y, z) = g(t, y - m, z)\) + comparison theorem.
Dual representation

\[-dY_t = g(t, Y_t, Z_t)dt - \langle Z_t, dW_t \rangle, \quad Y_T = -X_T\]

\[\Rightarrow \text{ Let } G(t, \beta, \mu) \text{ be the Fenchel transform of } g(t, y, z), \ (\beta \geq 0)\]

\[\Rightarrow \text{ Any } Q^\mu \text{ equivalent probability measure w.r. to } P\]

\[
\frac{dQ^\mu}{dP} := \Gamma_T^\mu, \quad d\Gamma_t^\mu = \Gamma_t^\mu \mu_t^* dW_t, \quad \Gamma_0^\mu = 1
\]

\[\Rightarrow D_T := e^{-\int_0^T \beta_s ds} \text{ discount factor } (\beta \geq 0)\]

\[\Rightarrow \text{ penalty functions } : G(t, \beta, \mu) := \mathbb{E}_{Q^\mu} \left[ \int_t^T e^{-\int_t^r \beta_s ds} G(\beta_r, \mu_r) dr \mid \mathcal{F}_t \right];\]

Then \( Y_t = \mathcal{R}_t^g \) has the following dual representation

\[
\mathcal{R}_t^g(X_T) = \text{ess sup}_{(\beta, \mu) \in \mathcal{A}} \mathbb{E}_{Q^\mu} \left[ -e^{-\int_t^T \beta_s ds} X_T - \int_t^T e^{-\int_t^r \beta_s ds} G(r, \mu) dr \mid F_t \right]
\]
Summary and conclusions

⇒ We introduce a new class of cash sub-additive risk measures $R$
⇒ $R$ suitable to assess the risk of financial positions under uncertain interest rates and other risks (e.g. default risk)
⇒ we extend $R$ to cash invariant r.m. defined on enlarged spaces
  • we extend $R$ to cash invariant r.m. defined on enlarged spaces
  • Dual representation of $R$
  • we identify $R$ from cash invariant r.m. in term of $F_T$-measures and $F_T$-stochastic discount factors