These notes provide a very brief introduction to pricing European options. This sketch is the latest version of a short introduction written for beginning quants at Commerzbank, written while consulting in Frankfurt. It also served, in modified form, as a brief introduction for students on the MSc in Mathematical Finance, when I was lecturing at Imperial College. The latest incarnation differs from these in that it’s based on Matlab.

1. A brief introduction to European options

These notes are fairly self-contained: some review of probability theory is discussed in a separate section, and background information is kept to a minimum. Further, there are many important points that are merely sketched here, but will be discussed in detail during the principal courses.

A European option is any function \( f \) that satisfies the equation

\[
f(S(t), t) = e^{-rt} \mathbb{E} f(S(t + h), t + h), \quad \text{for any } h > 0. \tag{1.1}
\]

Here \( r \) is the risk-free interest rate, which is the interest rate for a currency paid by a central bank. It’s not really risk-free, but is far more reliable than other investments, and we shall assume that it’s constant. The asset price \( S(t) \) evolves randomly according to a mathematical model called geometric Brownian motion. Its full definition is rather complicated, but the crucial equation is

\[
S(t + h) = S(t) \exp((r - \sigma^2/2)h + \sigma \sqrt{h} Z_t), \quad \text{for any } h > 0, \tag{1.2}
\]

where \( \sigma \) is a positive constant called the volatility of the asset. The randomness is provided by the random variable \( Z_t \), which is a normalized Gaussian random variable (i.e. mean zero and variance one). In particular, to generate sample prices \( S(T) \) at some future time \( T \) given the initial price \( S(0) \), we use

\[
S(T) = S(0) \exp((r - \sigma^2/2)T + \sigma \sqrt{T} Z_T), \quad \text{where } Z_T \sim N(0, 1). \tag{1.3}
\]
There are many important details omitted here, but we can learn a great deal by studying the mathematical consequences of (1.1) and (1.2). We see that (1.1) describes a contract $f(S, t)$ whose current value is the discounted value of its expected future value.

**Example 1.1.** A plain vanilla European put option is a European option for which the function $f_P(S, t)$ obeys the condition

$$f_P(S(T), T) = (K - S(T))_+, \quad (1.4)$$

where $T$ is called the expiry time of the option, the constant $K$ is called the exercise price, and $(z)_+ := \max\{z, 0\}$. This is simply an insurance contract that allows us to sell one unit of the asset at the exercise price $K$ at time $T$ in the future. If the asset’s price $S(T)$ is less than $K$ at this expiry time, then the option is worth $K - S(T)$, otherwise it’s worthless. Such contracts protect us if we’re worried that the asset’s price might drop.

Often, we know the value of the option $f(S(T), T)$ for all values of the asset $S(T)$ at some future time $T$. Our problem is to compute its value at some earlier time, because we’re buying or selling this option.

**Example 1.2.** A plain vanilla European call option is a European option for which the function $f_C(S, t)$ obeys the condition

$$f_C(S(T), T) = (S(T) - K)_+, \quad (1.5)$$

using the same notation as Example 1.1. This gives us the right to buy one unit of the asset at the exercise price $K$ at time $T$. If the asset’s price $S(T)$ exceeds $K$ at this expiry time, then the option is worth $S(T) - K$, otherwise it’s worthless. Such contracts protect us if we’re worried that the asset’s price might rise.

How do we compute $f(S(0), 0)$? The difficult part is computing the expected future value $E[f(S(T), T)]$. This can be done analytically for a tiny number of options, including the European Put and Call (see Theorem 1.2), but usually we must resort to a numerical calculation. This leads us to our first algorithm: Monte Carlo simulation. Here we choose a large integer $N$ and generate $N$ pseudo-random numbers $Z_1, Z_2, \ldots, Z_N$ that have the normalized Gaussian distribution; in Matlab, we simply write $Z = \text{randn}(N, 1)$. Using (1.2), these generate the future asset prices

$$S_k = S(0) \exp\left( (r - \frac{\sigma^2}{2})T + \sigma \sqrt{T} Z_k \right), \quad k = 1, \ldots, N. \quad (1.6)$$

We then approximate the future expected value by an average, that is, we take

$$f(S(0), 0) \approx \frac{e^{-rT}}{N} \sum_{k=1}^{N} f(S_k, T). \quad (1.7)$$
Monte Carlo simulation has the great advantage that it is extremely simple to program. Its disadvantage is that the error is usually a multiple of \( \frac{1}{\sqrt{N}} \), so that very large \( N \) is needed for high accuracy (each decimal place of accuracy requires about a hundred times more work). We note that (1.7) will compute the value of any European option that is completely defined by a known final value \( f(S(T), T) \).

We shall now use Monte Carlo to approximately evaluate the European Call and Put contracts. In fact, Put-Call parity, described below in Theorem 1.1, implies that we only need a program to calculate one of these, because they are related by the simple formula

\[
 f_C(S(0), 0) - f_P(S(0), 0) = S(0) - Ke^{-rT}. 
\]  

Here's the Matlab program for the European Put.

```matlab
%% These are the parameters chosen in Example 11.6 of
%% OPTIONS, FUTURES AND OTHER DERIVATIVES,
%% by John C. Hull (Prentice Hall, 4th edn, 2000)
%%
%% initial stock price
S0 = 42;
%% unit of time = year
%% 250 working days per year
%% continuous compounding risk-free rate
r = 0.1;
%% exercise price
K = 40;
%% time to expiration in years
T = 0.5;
%% volatility
sigma = 0.2;
%% generate asset prices at expiry
Z = randn(N,1);
ST = S0*exp( (r-(sigma^2)/2)*T + sigma*sqrt(T)*Z );
%% calculate put contract values at expiry
fput = max(K - ST,0.0);
%% average put values at expiry and discount to present
mc_put = exp(-r*T)*sum(fput)/N
%% calculate analytic value of put contract
wK = (log(K/S0) - (r - (sigma^2)/2)*T)/(sigma*sqrt(T));
a_put = K*exp(-r*T)*Phi(wK) - S0*Phi(wK - sigma*sqrt(T))
```

This program, and the utility function `Phi.m`, can be obtained from my homepage.
We have only revealed the tip of a massive iceberg in this brief introduction. Firstly, the Black-Scholes model, where asset prices evolve according to \((1.2)\), is rather poor: reality is far messier. Further, there are many types of option which are \textit{path-dependent}: the value of the option at expiry depends not only on the final price \(S(T)\), but on its previous values \(\{S(t): 0 \leq t \leq T\}\). Further, there are \textit{American options}, where the contract can be exercised at any time before its expiry. All of these points will be addressed in our course, but you should find that Hull’s book provides excellent background reading (although his mathematical treatment is often sketchy).

1.1. \textit{Analytic Values of European Puts and Calls}

It’s not too hard to calculate the values of these options analytically. Further, the next theorem gives an important relation between the prices of call and put options.

\textbf{Theorem 1.1. (Put-Call parity)} European Put and Call options satisfy

\[
f_C(S(0),0) - f_P(S(0),0) = S(0) - Ke^{-rT}.
\] (1.9)

\textit{Proof.} The trick is the observation that

\[
y = y_+ - (-y)_+,
\]

for any \(y \in \mathbb{R}\). Thus

\[
S(T) - K = f_C(S(T),T) - f_P(S(T),T),
\]

which implies

\[
e^{-rT} \left( E S(T) - K \right) = f_C(S(0),0) - f_P(S(0),0).
\]

Now

\[
E S(T) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}w} e^{-w^2/2} dw
\]

\[
= S(0)e^{(r-\sigma^2/2)T} (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(w^2-2\sigma\sqrt{T}w)} dw
\]

\[
= S(0)e^{rT},
\]

and some simple algebraic manipulation completes the proof. \(\square\)

This is a useful check on the Monte Carlo approximations of the options’ values. To derive their analytic values, we shall need the function

\[
\Phi(y) = (2\pi)^{-1/2} \int_{-\infty}^{y} e^{-z^2/2} dz, \quad y \in \mathbb{R},
\] (1.10)
which is simply the cumulative distribution function of the Gaussian probability density, that is, \( \mathbb{P}(Z \leq y) = \Phi(y) \) and \( \mathbb{P}(a \leq Z \leq b) = \Phi(b) - \Phi(a) \), for any normalized Gaussian random variable \( Z \).

**Theorem 1.2.** A European Put option satisfies

\[
f_P(S(0), 0) = Ke^{-rT} \Phi(w(K)) - S(0)\Phi(w(K) - \sigma\sqrt{T}), \tag{1.11}
\]

where \( w(K) \) is defined by the equation

\[
K = S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}w(K)},
\]

that is

\[
w(K) = \frac{\log(K/S(0)) - (r-\sigma^2/2)T}{\sigma\sqrt{T}}. \tag{1.12}
\]

**Proof.** We have

\[
\mathbb{E}_P(S(T), T) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \left( K - S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}w} \right) e^{-w^2/2} dw.
\]

Now the function

\[
w \mapsto K - S(0)\exp((r-\sigma^2/2)T + \sigma\sqrt{T}w)
\]

is strictly decreasing, so that

\[
K - S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}w} \geq 0
\]

if and only if \( w \leq w(K) \), where \( w(K) \) is given by (1.12). Hence

\[
\mathbb{E}_P(S(T), T) = (2\pi)^{-1/2} \int_{-\infty}^{w(K)} \left( K - S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}w} \right) e^{-w^2/2} dw
\]

\[
= K\Phi(w(K)) - S(0)e^{(r-\sigma^2/2)T}(2\pi)^{-1/2} \int_{-\infty}^{w(K)} e^{-1/2(w^2-2\sigma\sqrt{T}w)} dw
\]

\[
= K\Phi(w(K)) - S(0)e^{rT}\Phi(w(K) - \sigma\sqrt{T}).
\]

Discounting this expectation to its present value, we derive the option price.

**Exercise 1.1.** Modify the proof of this theorem to derive the analytic price of a European Call option. Check that your price agrees with Put-Call parity.

### 1.2. The Black-Scholes Equation

We can also use (1.1) to derive the famous **Black-Scholes partial differential equation**, which is satisfied by any European option. The key is to choose
a small positive $h$ in (1.1) and expand. We shall need Taylor’s theorem for functions of two variables, which states that

$$G(x + \delta x, y + \delta y) = G(x, y) + \left( \frac{\partial G}{\partial x} \delta x + \frac{\partial G}{\partial y} \delta y \right)$$

$$+ \frac{1}{2} \left( \frac{\partial^2 G}{\partial x^2} (\delta x)^2 + 2 \frac{\partial^2 G}{\partial x \partial y} (\delta x)(\delta y) + \frac{\partial^2 G}{\partial y^2} (\delta y)^2 \right) + \cdots. \tag{1.13}$$

Further, it simplifies matters to use “log-space”: we introduce $\tilde{S}(t) := \log S(t)$, where $\log = \log_e$ in these notes (not logarithms to base 10). In log-space, (1.2) becomes

$$\tilde{S}(t + h) = \tilde{S}(t) + (r - \sigma^2) h + \sigma h^{1/2} Z_t. \tag{1.14}$$

We also introduce

$$g(\tilde{S}(t), t) := f(\exp(\tilde{S}(t), t)), \tag{1.15}$$

so that (1.1) takes the form

$$g(\tilde{S}(t), t) = e^{-rh} \mathbb{E}g(\tilde{S}(t + h), t + h). \tag{1.16}$$

Now Taylor expansion yields the (initially daunting)

$$g(\tilde{S}(t + h), t + h) = g(\tilde{S}(t), t + h) + \frac{\partial g}{\partial S} \left( (r - \sigma^2/2) h + \sigma h^{1/2} Z_t \right) +$$

$$+ \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \sigma^2 h Z_t^2 + h \frac{\partial g}{\partial t} + \cdots, \tag{1.17}$$

ignoring all terms of higher order than $h$. Further, since $\mathbb{E}Z_t = 0$ and $\mathbb{E}(Z_t^2) = 1$, we obtain

$$\mathbb{E}g(\tilde{S}(t + h), t + h) = g(\tilde{S}(t), t) + h \left( \frac{\partial g}{\partial S} (r - \sigma^2/2) + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \sigma^2 + \frac{\partial g}{\partial t} \right) + \cdots. \tag{1.18}$$

Recalling that

$$e^{-rh} = 1 - rh + \frac{1}{2} (rh)^2 + \cdots,$$

we find

$$g = (1 - rh + \cdots) \left( g + h \left[ \frac{\partial g}{\partial S} (r - \sigma^2/2) + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \sigma^2 + \frac{\partial g}{\partial t} \right] + \cdots \right)$$

$$= g + h \left( -rg + \frac{\partial g}{\partial S} (r - \sigma^2/2) + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \sigma^2 + \frac{\partial g}{\partial t} \right) + \cdots. \tag{1.19}$$
For this to be true for all $h > 0$, we must have
\[-rg + \frac{\partial g}{\partial S}(r - \sigma^2/2) + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \sigma^2 + \frac{\partial g}{\partial t} = 0,\] (1.20)
and this is the celebrated Black-Scholes partial differential equation (PDE).

Thus, instead of computing an expected future value, we can calculate the solution of the Black-Scholes PDE (1.20). The great advantage gained is that there are highly efficient numerical methods for solving PDEs numerically. The disadvantages are complexity of code and learning the mathematics needed to exploit these methods effectively.

1.3. Asian Options

European Put and Call options provide a useful laboratory in which to understand and test methods. However, the main aim of Monte Carlo is to calculate option prices for which there is no convenient analytic formula. We shall illustrate this with Asian options. Specifically, we shall consider the option
\[f_A(S, T) = \mu \left( S(T) - \frac{1}{T} \int_0^T S(\tau) \, d\tau \right) + .\] (1.21)

This is a path dependent option: its value depends on the history of the asset price, not simply its final value. (NB: I have changed this Asian option from that considered in earlier versions of the notes.)

Why would anyone trade Asian options? Consider a bank’s corporate client trading in, say, Britain and the States. The client’s business is exposed to exchange rate volatility: the pound’s value in dollars varies over time. Therefore the client may well decide to hedge by buying an option to trade dollars for pounds at a set rate at time $T$. This can be an expensive contract for the writer of the option, because currency values can “blip”. An alternative contract is to make the exchange rate at time $T$ a time-average, as in (1.21). Any contract containing time-averages of asset prices is usually called an Asian option, and there are many variants of these. For example, the option dual to (1.21) (in the sense that a call option is dual to a put option) is given by
\[g_A(S, T) = \left( \frac{1}{T} \int_0^T S(\tau) \, d\tau - S(T) \right) + .\] (1.22)

Pricing (1.21) via Monte Carlo is fairly simple. We choose a positive integer $M$ and subdivide the time interval $[0, T]$ into $M$ equal subintervals. We evolve the asset price using the equation
\[S\left( \frac{(k + 1)T}{M} \right) = S\left( \frac{kT}{M} \right) e^{(r - \sigma^2/2) \frac{T}{M} + \sigma \sqrt{\frac{T}{M}} Z_k}, \quad k = 0, 1, \ldots, M - 1,\] (1.23)
where $Z_0, Z_1, \ldots, Z_{M-1}$ are independent $N(0, 1)$ independent pseudorandom numbers generated by Box-Muller. We could approximate the time-average integral by the trapezium rule
\[
\int_0^T S(\tau) \, d\tau \approx \frac{1}{M} \left( \frac{1}{2} S(0) + \frac{1}{2} S(T) + \sum_{k=1}^{M-1} S\left( \frac{kT}{M} \right) \right),
\]
giving a discrete approximation $\tilde{f}_A$. We might even use the simpler approximation
\[
T^{-1} \int_0^T S(\tau) \, d\tau \approx M^{-1} \sum_{k=0}^{M-1} S\left( \frac{kT}{M} \right).
\]

**Exercise 1.2.** Write a program to price the discrete Asian option defined by
\[
f_M(S,T) = \left( S(T) - M^{-1} \sum_{k=0}^{M-1} S\left( \frac{kT}{M} \right) \right)^+. \tag{1.24}
\]

1.4. **Probability Theory**

A random variable $X$ is said to have (continuous) probability density function $p(t)$ if
\[
\mathbb{P}(a < X < b) = \int_a^b p(t) \, dt. \tag{1.25}
\]
We shall assume that $p(t)$ is a continuous function (no jumps in value). In particular, we have
\[
1 = \mathbb{P}(X \in \mathbb{R}) = \int_{-\infty}^{\infty} p(t) \, dt.
\]
Further, because
\[
0 \leq \mathbb{P}(a < X < a + \delta a) = \int_a^{a+\delta a} p(t) \, dt \approx p(a) \delta a,
\]
for small $\delta a$, we conclude that $p(t) \geq 0$, for all $t \geq 0$. In other words, a probability density function is simply a non-negative function $p(t)$ whose integral is one. Here are two fundamental examples.

**Example 1.3.** The Gaussian probability density function, with mean $\mu$ and variance $\sigma^2$, is defined by
\[
p(t) = (2\pi\sigma^2)^{-1/2} \exp\left( -\frac{(t-\mu)^2}{2\sigma^2} \right). \tag{1.26}
\]
We say that the Gaussian is *normalized* if $\mu = 0$ and $\sigma = 1$. 
To prove that this is truly a probability density function, we require the important identity
\[ \int_{-\infty}^{\infty} e^{-Cx^2} \, dx = \sqrt{\pi/C}, \] (1.27)
which is valid for any \( C > 0 \). [In fact it’s valid for any complex number \( C \) whose real part is positive.]

**Example 1.4.** The Cauchy probability density function is defined by
\[ p(t) = \frac{1}{\pi(1+t^2)}. \] (1.28)
This distribution might also be called the *Mad Machine Gunner* distribution. Imagine our killer sitting at the origin of the \((x, y)\) plane. He is firing (at a constant rate) at the infinite line \( y = 1 \), his angle \( \theta \) (with the \( x \)-axis) of fire being uniformly distributed in the interval \((0, \pi)\). Then the bullets have the Cauchy density.

If you draw some graphs of these probability densities, you should find that, for small \( \sigma \), the graph is concentrated around the value \( \mu \). For large \( \sigma \), the graph is rather flat. There are two important definitions that capture this behaviour mathematically.

**Definition 1.1.** The mean, or expected value, of a random variable \( X \) with p.d.f \( p(t) \) is defined by
\[ \mathbb{E}X := \int_{-\infty}^{\infty} tp(t) \, dt. \] (1.29)
It’s very common to write \( \mu \) instead \( \mathbb{E}X \) when no ambiguity can arise. Its variance \( \text{Var} \ X \) is given by
\[ \text{Var} \ X := \int_{-\infty}^{\infty} (t - \mu)^2 p(t) \, dt. \] (1.30)

**Exercise 1.3.** Show that the Gaussian p.d.f. really does have mean \( \mu \) and variance \( \sigma^2 \).

**Exercise 1.4.** Find the mean and variance of the Cauchy probability density defined in Example 1.28.

**Exercise 1.5.** Prove that \( \text{Var} \ X = \mathbb{E}(X^2) - (\mathbb{E}X)^2 \).

We shall frequently have to calculate the expected value of functions of random variables.

**Theorem 1.3.** If
\[ \int_{-\infty}^{\infty} |f(t)|p(t) \, dt \]
is finite, then
\[ E(f(X)) = \int_{-\infty}^{\infty} f(t)p(t) \, dt. \] (1.31)

**Example 1.5.** Let \( X \) denote a normalized Gaussian random variable. We shall show that
\[ E(e^{\lambda X}) = e^{\lambda^2/2}, \] (1.32)
Indeed, applying (1.31), we have
\[ E(e^{\lambda X}) = \int_{-\infty}^{\infty} e^{\lambda t} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2 - 2\lambda t)} \, dt. \]
The trick now is to complete the square in the exponent, that is,
\[ t^2 - 2\lambda t = (t - \lambda)^2 - \lambda^2. \]
Thus
\[ E(e^{\lambda X}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp\left( -\frac{1}{2} [(t - \lambda)^2 - \lambda^2]\right) \, dt = e^{\lambda^2/2}. \]

**Exercise 1.6.** Calculate \( E(\cosh X) \) for a Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \). [Here \( \cosh t = (e^t + e^{-t})/2 \).]

1.5. Mortgages – a once exotic instrument

The objective of this section is to remind you of some basic facts regarding difference and differential equations via mortgage pricing. You are presumably all too familiar with a repayment mortgage: we borrow a large sum \( M \) for a fairly large slice \( T \) of our lifespan, repaying capital and interest using \( N \) regular payments. Younger readers may be surprised to learn that it was once possible to buy property in London. The interest \( e^r \), which we shall assume to be constant, is not much larger than the risk-free interest rate, because our homes are forfeit on default. How do we calculate our repayments?

Let \( h = T/N \) be the interval between payments, let \( D_h : [0, T] \to \mathbb{R} \) be our debt as a function of time, and let \( A(h) \) be our payment. We shall assume that our initial debt is \( D_h(0) = 1 \), because we can always multiply by the true initial cost \( M \) of our house after the calculation. Thus \( D_h \) must satisfy the equations
\[ D_h(0) = 1, \quad D_h(T) = 0 \quad \text{and} \quad D_h((\ell - 1)h) = D_h((\ell - 1))e^{rh} - A(h). \quad (1.33) \]

**Exercise 1.7.** Show that
\[ D_h((\ell - 1)h) = e^{\ell rh} - A(h) \left( \frac{e^{\ell rh} - 1}{e^{rh} - 1} \right). \quad (1.34) \]
Deduce that
\[ A(h) = \frac{e^{rh} - 1}{1 - e^{-rT}}, \]  
(1.35)

What happens if \( T \to \infty? \)

Almost all mortgages are repaid by 300 monthly payments for 25 years. However, many mortgages calculate interest yearly, which means that we choose \( h = 1 \) in Exercise 1.7 and then divide \( A(1) \) by 12.

**Exercise 1.8.** Calculate the monthly repayment \( A(1) \) when \( M = 10^5 \), \( T = 25 \), \( r = 0.05 \) and \( h = 1 \). Now repeat the calculation using \( h = 1/12 \). Interpret your result.

In principle, there’s no reason why our repayment could not be continuous, with interest being recalculated on our constantly decreasing debt. For continuous repayment, our debt \( D : [0, T] \to \mathbb{R} \) satisfies the relations
\[ D(0) = 1, \quad D(T) = 0 \quad \text{and} \quad D(t + h) = D(t)e^{rh} - hA. \]  
(1.36)

**Exercise 1.9.** Prove that
\[ D'(t) - rD(t) = -A, \]  
(1.37)
where, in particular, you should prove that (1.36) implies the differentiability of \( D(t) \). Solve this differential equation using the integrating factor \( e^{-rt} \).

You should find the solution
\[ D(t)e^{-rt} - 1 = A \int_0^t (-e^{-rt}) \, d\tau = A \left( \frac{e^{-rt} - 1}{r} \right). \]  
(1.38)
Hence show that
\[ A = \frac{r}{1 - e^{-rT}} \]  
(1.39)
and
\[ D(t) = \frac{1 - e^{-r(T-t)}}{1 - e^{-rT}}. \]  
(1.40)
Prove that \( \lim_{r \to \infty} D(t) = 1 \) for \( 0 < t < T \) and interpret.

Observe that
\[ \frac{A(h)}{Ah} = \frac{e^{rh} - 1}{rh} \approx 1 + (rh/2), \]  
(1.41)
so that continuous repayment is optimal for the borrower. What’s the profit for the lender?

**Exercise 1.10.** Construct graphs of \( D(t) \) for various values of \( r \). Calculate the time \( t_0(r) \) at which half of the debt has been paid.